Semestral

Class Field Theory

Instructor: Ramdin Mawia **Marks:** 50 **Time:** May 02, 2023; 10:00–13:00.

INSTRUCTIONS

i. Attempt THREE problems, including problem n° 5. The marks are indicated against each question.

ii. You may use any of the results proved in class, unless you are asked to prove or justify the result itself. You may also use results from other problems in this question paper, provided you attempt and correctly solve the problem.

iii. The notation is standard: $\mathbb{A}_K$ denotes the ring of adèles of a global field K, and $\mathbb{A}_K^\times$ denotes its group of idèles. We write $\mathbb{A}_{K,f}^\times$ for the finite idèles, that is, the restricted product of $K_v^\times$ with respect to $\mathcal{O}_v^\times$ for v finite.

CLASS FIELD THEORY

1. State the Kronecker-Weber theorem, and prove it using the main results of global class field theory, clearly stating all the results you use. Try to be as self-contained and complete as possible. **20**

2. Let L/K be a finite extension of nonarchimedean local fields of characteristic 0, and let E be the largest abelian extension of K contained in L. Prove that $\mathrm{N}_{L/K}L^\times = \mathrm{N}_{E/K}E^\times$. [*Hint.* For L/K Galois, use Local Artin Reciprocity. Else look at its Galois closure and use functoriality.] **20**

3. Let p be an odd prime and $\zeta_p \in \mathbb{C}$ be a primitive p^{th} root of 1. Let $L = \mathbb{Q}[\zeta_p]$. **20**

   i. Show that $K = \mathbb{Q}[\sqrt{\epsilon_p p}]$ is the unique quadratic number field contained in L, where $\epsilon_p = (-1)^{(p-1)/2}$.

   ii. Let ℓ be an odd prime distinct from p. Show that
   $$\varphi_{K/\mathbb{Q}}(\ell) = \left(\frac{\epsilon_p p}{\ell}\right) \qquad (*)$$
   where $\varphi_{K/\mathbb{Q}} : \mathbb{A}_\mathbb{Q}^\times \to \mathrm{Gal}(K/\mathbb{Q}) \cong \{\pm 1\}$ is the Global Artin Reciprocity Map for $K/\mathbb{Q}$, and $(\cdot/\ell)$ on the right of (*) stands for the Legendre symbol mod ℓ. Also, ℓ in the argument of $\varphi_{K/\mathbb{Q}}$ represents the idèle $(1, \ldots, 1, \ell, 1, \ldots, 1)$ under the usual embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{A}_\mathbb{Q}^\times$.

   iii. Conclude that
   $$\left(\frac{p}{\ell}\right)\left(\frac{\ell}{p}\right) = (-1)^{\frac{p-1}{2}\frac{\ell-1}{2}}.$$

4. Let E/F be a finite, unramified extension of nonarchimedean local fields of characteristic 0. Show that $\mathrm{N}_{E/F}\mathcal{O}_E^\times = \mathcal{O}_F^\times$. Here $\mathcal{O}_E^\times$ and $\mathcal{O}_F^\times$ denote the unit groups of E and F respectively. **20**

5. State true or false, with brief but complete justifications (**any four**): **10**

   i. For a number field K, every open subgroup of the idèle class group $\mathbf{C}_K := K^\times \backslash \mathbb{A}_K^\times$ is of finite index.

   ii. For a local field F of characteristic 0, every subgroup $H < F^\times$ of finite index is open in $F^\times$.

   iii. Let $\omega \in \mathbb{C}$ be a primitive 2023^{rd} root of unity and let $K = \mathbb{Q}[\omega]$. Then the unit group $\mathcal{O}_K^\times$ of the number field K is a finitely generated abelian group of rank 2022.

   iv. Let $K = \mathbb{Q}[\sqrt{2}]$. Then $K^\times$, diagonally embedded, is a closed subgroup of the finite idèles $\mathbb{A}_{K,f}^\times$.

   v. Let ℓ/k be a finite extension of finite fields. Then the norm morphism $\mathrm{N}_{\ell/k} : \ell^\times \to k^\times$ is surjective.